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Properties of Sompolinsky's mean field theory of spin glasses

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Abstract. We derive certain properties of Sompolinsky's mean field theory of spin glasses showing its consistency. We obtain an exact relation which replaces the Parisi–Toulouse hypothesis. Parisi's parameter X is identified with an averaged squared 3-spin vertex. We expand the solution along the de Almeida–Thouless line using a Parisi-type differential equation for the free energy.

1. Introduction

The spin glass model of Sherrington and Kirkpatrick (1975) can be solved exactly using the replica method or the method of diagram expansion (Sommers 1978) provided that the temperature exceeds a critical value which for a finite external field is determined by the de Almeida–Thouless (1978) instability line. Below the spin glass transition so-called solutions can be produced by breaking the replica symmetry. The most successful replica solution is that of Parisi (1979). Replica symmetry breaking is necessarily connected with 'breaking of the linear response', which means that due to the interaction the dependence of the local spin expectation value on the local external field becomes effectively more complicated than in the paramagnetic phase. Below the spin glass transition the diagram expansion breaks down since certain correlation functions do not have the usual clustering property.

For instance, for $N \to \infty$, $i \neq j$:

$$\overline{\langle S_i S_j \rangle^2} > \overline{\langle S_i \rangle^2}\,\overline{\langle S_j \rangle^2}. \tag{1}$$

The left-hand side determines just the internal energy (Bray and Moore 1980). We are considering a model of N interacting spins $S_i = \pm 1$ in local external fields b_i with the Hamiltonian

$$\mathcal{H} = -\sum_{i<j} J_{ij} S_i S_j - \sum_i b_i S_i. \tag{2}$$

The brackets $\langle \ldots \rangle$ denote the thermal average with the weight $\exp(-\beta\mathcal{H})$. The bar $\overline{\ldots}$ denotes the average with respect to the long-range random interactions having a probability distribution

$$P[J_{ij}] \propto \exp\Big(-\tfrac{1}{2} \sum_{i<j} J_{ij}^2 N/J^2\Big). \tag{3}$$

Parisi introduced one order parameter function $q(X)$, $0 \leq X \leq 1$, but the meaning of the parameter X has been mysterious. It has been suggested that $q(1)$ is connected with the static Edwards–Anderson order parameter $q=\overline{\langle S_i\rangle^2}$ (Parisi 1980a, Thouless *et al* 1980). Young and Kirkpatrick (1982) pointed out that this interpretation cannot be correct because of inequality (1). Actually it will be published elsewhere that $q=q(0)$. This coincides with an interpretation of the variable X which has been given by Sompolinsky (1981). Sompolinsky and Zippelius (1981) consider a dynamic version of the model. From this Sompolinsky (1981) derives an expression for the free energy which is stationary with respect to two order parameter functions $q(X)$ and $\Delta(X)$. $q(X)$ is connected with an averaged local spin–spin correlation on a timescale τ_X:

$$q(X)=\overline{\langle S_i S_i(\tau_X)\rangle}. \tag{4}$$

The timescales are arranged in decreasing order, but all go to infinity in the thermodynamic limit. $\beta(1-q(1)+\Delta(X))$ is the averaged local susceptibility on the timescale τ_X. Thus $q(X)$ is connected with the time-persistent part of the spin–spin correlation. This idea is originally due to Edwards and Anderson (1975). However, there exists a whole spectrum of limits corresponding to an incomplete decay of the local dynamical spin–spin correlation. Our result $q(0)=q$ coincides with this interpretation. It means that on the largest timescale the decay of the dynamical correlation is complete. $q(1)$ is the spin–spin correlation on the lowest timescale; it can be calculated by Monte Carlo computer simulations (Kirkpatrick and Sherrington 1978, Parisi 1980b). On the timescale τ_1 linear response theory is valid; thus $\Delta(1)=0$. Fischer's (1976) relation for the local susceptibility, $\chi(1)=\beta(1-q(1))$, is valid, however, with $q(1)$ instead of q. The parameter $\Delta(X)$ measures the violation of the linear response on the timescale τ_X. According to their interpretations $q(X)$ is an increasing and $\Delta(X)$ a decreasing function of X, which shall be proved in the following. A complete derivation on a purely static basis of either the Parisi theory or the Sompolinsky theory is still lacking. Sompolinsky's theory can also be derived by the replica method (de Dominicis *et al* 1982). In this paper we shall derive some properties of the Sompolinsky mean field theory showing its consistency. Furthermore we will prove an exact relation for the local susceptibility which replaces the Parisi–Toulouse hypothesis. It has been pointed out by Sompolinsky and proved by de Dominicis *et al* (1982) that Parisi's solution is obtained if we set

$$\Delta'(X)=-Xq'(X). \tag{5}$$

The theory of Sompolinsky is invariant under a general scale transformation of X on the interval $(0,1)$. Equation (5) means simply a choice of a new scale if $\xi(X)=-\Delta'(X)/q'(X)$ is a monotonic function between 0 and 1. We express $\xi(X)$ generally by a 3-spin vertex. Since this vertex determines the interaction, the monotony of $\xi(X)$ corresponds to a decrease of dynamic correlations with time.

2. Properties of Sompolinsky's theory

Let us start with the Sompolinsky free energy functional

$$-\beta f=\frac{(\beta J)^2}{4}\left((1-q(1))^2+2\int_0^1 \mathrm{d}X\ \Delta'(X)q(X)\right)$$
$$+\overline{\log(2\cosh\beta H)+\frac{(\beta J)^2}{2}\int_0^1 \mathrm{d}X\ \Delta'(X)\{m\}_X^2} \tag{6}$$

with

$$\beta H = \beta b + \beta J z_0 \sqrt{q(0)} + \beta J \int_0^1 \mathrm{d}X\, z(X)\sqrt{q'(X)} - (\beta J)^2 \int_0^1 \mathrm{d}X\, \Delta'(X)\{m\}_X \tag{7}$$

and

$$m = \tanh \beta H. \tag{8}$$

$\overline{\ldots}$ means a symmetric Gaussian functional integral with

$$\overline{z(X)z(X')} = \delta(X - X') \qquad \overline{z_0^2} = 1. \tag{9}$$

$\{\ldots\}_X$ means averaging only over variables $z(X')$ with $X' > X$. $-\beta f$ is stationary with respect to variations of $q(0)$, $q'(X)$, $\Delta'(X)$ and $\{m\}_X$ as functional of $z(X')$ for $X' < X$. The self-consistency equations are

$$-\frac{\partial \beta f}{\partial q(0)} = \frac{(\beta J)^2}{2}\left(q(1) - 1 - \Delta(0) + \overline{\frac{\partial m}{\partial \beta b_0}}\right) = 0 \tag{10}$$

with $\beta b_0 = \beta J z_0 \sqrt{q(0)}$,

$$-\frac{\delta \beta f}{\delta q'(X)} = \frac{(\beta J)^2}{2}\left(q(1) - 1 - \Delta(X) + \overline{\frac{\delta m}{\delta \beta b(X)}}\right) = 0 \tag{11}$$

with $\beta b(X) = \beta J z(X)\sqrt{q'(X)}$, and

$$-\frac{\delta \beta f}{\delta \Delta'(X)} = \frac{(\beta J)^2}{2}\left(q(X) - \overline{\{m\}_X^2}\right) = 0. \tag{12}$$

We have added the term with $q(0)$ to Sompolinsky's expression for βH since in a finite external field, b, $\overline{\{m\}_0^2}$ is certainly not equal to 0. Equation (10) is the same as equation (11) for $X \to 0$. $\Delta(1) \stackrel{\text{def}}{=} 0$. If we add a term $(\beta J)^2 \Delta(1) m$ to βH and a corresponding term to $-\beta f$ we can make $-\beta f$ stationary with respect to $\Delta(1)$; however, a non-trivial solution (unequal to Sommers's (1978) solution) is obtained only for $\Delta(1) = 0$.

From equations (7) and (8) we find the following functional integral equation for $\delta m / \delta \beta b(X)$:

$$\frac{\delta m}{\delta \beta b(X)} = (1 - m^2)\left(1 - (\beta J)^2 \int_X^1 \mathrm{d}X'\, \Delta'(X') \left\{\frac{\delta m}{\delta \beta b(X)}\right\}_{X'}\right). \tag{13}$$

From this it is immediately seen that $\delta m / \delta \beta b(X) \geqslant 0$ for $\Delta'(X) \leqslant 0$. Furthermore

$$\frac{\delta m}{\delta \beta b(1)} = 1 - m^2 \qquad \frac{\delta m}{\delta \beta b(0)} = \frac{\partial m}{\partial \beta b_0}. \tag{14}$$

Let us differentiate equation (12) with respect to X:

$$q'(X) = \lim_{\delta \to 0} \overline{(\{m\}_{X+\delta}^2 - \{m\}_X^2)}/\delta = \lim_{\delta \to 0} \overline{(\{m\}_{X+\delta} - \{m\}_X)^2}/|\delta|. \tag{15}$$

Thus $q'(X)$ is self-consistently non-negative. If we expand m in powers of

$\int_X^{X+\delta} \beta b(X)\, dX$ we get, in the limit $\delta \to 0$,

$$q'(X) = \lim_{\delta\to 0} \overline{\left\{\frac{\delta m}{\delta\beta b(X)}\right\}_X^2 \left(\int_X^{X+\delta} dX\, \beta J z(X)\sqrt{q'(X)}\right)^2} \Big/ |\delta|$$
$$= q'(X)(\beta J)^2 \overline{\left\{\frac{\delta m}{\delta\beta b(X)}\right\}_X^2}. \tag{16}$$

This kind of differentiation is rather tricky, since we always have to expand up to second order of the integral to get a contribution proportional to δ in the average.

Let us introduce the notation

$$\Gamma_n(X) = \left\{\left(\frac{\delta}{\delta\beta b(X)}\right)^{n-1} m\right\}_X. \tag{17}$$

Then the self-consistency equations (11) and (12) are

$$1 - q(1) + \Delta(X) = \overline{\Gamma_2(X)} \qquad q(X) = \overline{\Gamma_1(X)^2}.$$

Equation (16) reads now

$$q'(X)[1 - (\beta J)^2 \overline{\Gamma_2(X)^2}] = 0.$$

This equation means $q'(X) = 0$ or

$$1 = (\beta J)^2 \overline{\Gamma_2(X)^2}. \tag{18}$$

Thus for $X = 1$

$$1 = (\beta J)^2 \overline{(1 - m^2)^2} \tag{19}$$

which is just the Bray–Moore (1979) condition for a soft mode. However, remember that this equation is not valid for the equilibrium expectation values: $m \neq \langle S_i \rangle$. Instead for $X \to 0$ we have

$$1 = \overline{\left(J\frac{\partial}{\partial b_0}\{m\}_0\right)^2} \tag{20}$$

which together with the information $q(0) \to 0$ for $b \to 0$ gives the constant susceptibility for $b \to 0$. However, equation (20) is correct for all external fields and represents the correct formulation of the Parisi–Toulouse (1980) hypothesis. According to Sompolinsky and Zippelius (1981), equation (19) means marginal dynamic stability. We have found that an analogous equation is valid on each timescale provided that $q'(X) \neq 0$. Let us now consider the second self-consistency equation (11) and derivate it with respect to X. From equation (13) we get

$$\frac{\partial}{\partial X}\frac{\delta m}{\delta\beta b(X)} = (\beta J)^2 \Delta'(X)\Gamma_2(X)(1 - m^2) - (1 - m^2)\int_X^1 dX'\, \Delta'(X')\left\{\frac{\partial}{\partial X}\frac{\delta m}{\delta\beta b(X)}\right\}_{X'}. \tag{21}$$

Comparing equations (13) and (21) we find

$$\frac{\partial}{\partial X}\frac{\delta m}{\delta\beta b(X)} = (\beta J)^2 \Delta'(X)\Gamma_2(X)\frac{\delta m}{\delta\beta b(X)} \tag{22}$$

since $\Gamma_2(X)$ does not depend on $z(X')$ for $X' > X$. Derivating now equation (11) with respect to X we find

$$\Delta'(X) = (\beta J)^2 \overline{\Gamma_2(X)^2} \Delta'(X). \tag{23}$$

Thus equation (18) is valid if $\Delta'(X)$ and $q'(X)$ are not equal to zero simultaneously. Assume now that $\Delta'(X)$ and $q'(X)$ are identically equal to 0 in a whole interval. Then, since this interval does not contribute to the functional integral at all, we may define new continuous functions, $\Delta(X)$ and $q(X)$, eliminating the flat region and rescaling X on the interval (0, 1) without changing $-\beta f$. Therefore either both $\Delta'(X)$ and $q'(X) \equiv 0$ (Sherrington–Kirkpatrick solution) or we may assume that $q'(X)$ and $\Delta'(X)$ are monotonic functions never constant on any sub-interval of (0, 1). We shall see soon that $q'(X)$ and $\Delta'(X)$ cannot be zero separately. Note that, together with the information $\Delta'(X) \not\equiv 0, q'(X) \not\equiv 0, \Delta(1) = 0$, both self-consistency equations (11) and (12) are equivalent to equation (18) with the initial condition

$$q(1) = \overline{m^2}. \tag{24}$$

Instead of equation (24) we may also choose

$$q(0) = \overline{\{m\}_0^2} \tag{25}$$

since equation (24) follows then by integration of equation (16). For $b \to 0$ equation (25) obviously has the solution $q(0) = 0$. Then equation (18) is the only independent self-consistency equation. From equation (20) the constant susceptibility for $b \to 0$ follows.

We try now to get further information by differentiating equation (18) with respect to X. The result is

$$0 = q'(X) \overline{\Gamma_3(X)^2} + 2\Delta'(X) \overline{\Gamma_2(X)^3}. \tag{26}$$

The first term follows in analogy to equation (16), the second using equation (22). Both coefficients are positive: thus, if $q'(X) = 0$, then $\Delta'(X) = 0$ too and vice versa. Furthermore $\Delta'(X)$ is self-consistently non-positive. Comparing with Parisi's X, equation (5), we have

$$\xi(X) = X_{\text{Parisi}} = \tfrac{1}{2} \overline{\Gamma_3(X)^2} / \overline{\Gamma_2(X)^3}. \tag{27}$$

$\xi(X)$ varies between

$$\xi(0) = \frac{1}{2\beta} \overline{\left(\frac{\partial^2}{\partial b_0{}^2} \{m\}_0 \right)^2} \Big/ \overline{\left(\frac{\partial}{\partial b_0} \{m\}_0 \right)^3} \tag{28}$$

and

$$\xi(1) = 2\overline{m^2(1-m^2)^2} / \overline{(1-m^2)^3}. \tag{29}$$

Note that the last parameter just determines the critical index for the low-frequency behaviour of the dynamical susceptibility according to the theory of Sompolinsky and Zippelius (1981). The monotonous increase of $\xi(X)$ corresponds to a decrease of dynamic correlations with time. We can calculate $\xi'(X)$ using the above method, which can be seen to be positive in certain limiting cases; however, we did not succeed in a general proof.

Assuming the monotony, we can choose $\xi(X)$ as a new scale. It is not yet normalised to the interval (0, 1). If we extend the solution constant outside the interval ($\xi(0)$, $\xi(1)$) (which does not affect the free energy) we get exactly the Parisi result. The stability analysis of Thouless *et al* (1980) is valid since indeed $q'(X) \geq 0$; the flat regions in Parisi's solution are not of physical importance. It remains also to show that $\xi(X)$ is actually smaller than 1. Since it is maximal for $X=1$ we have to show using equation (29)

$$\overline{(1-3m^2)(1-m^2)^2}>0. \tag{30}$$

At least along the de Almeida–Thouless instability line we can show the inequality. There the left-hand side goes to 1 for $T\to T_c$ $(b\to 0)$ and to $\frac{2}{5}$ for $T\to 0$ $(b\to\infty)$. Expansion along the instability line shows that condition (30) ensures that $\Delta(0)$ never becomes singular along the whole instability line.

3. Expansion near the transition

Now we derive a Parisi differential equation for the Sompolinsky free energy functional. This has already been done with the help of the replica method by de Dominicis *et al* (1982). We may write

$$-\beta f=\frac{(\beta J)^2}{4}\Big((1-q(1))^2+2\int_0^1 \mathrm{d}X\ \Delta'(X)q(X)\Big)+\int\frac{\mathrm{d}z}{\sqrt{2\pi}}\,\mathrm{e}^{-z^2/2}\phi(\beta b+z\beta J\sqrt{q(0)},0) \tag{31}$$

where $\phi(\beta b, X)$ is equal to the corresponding expression in equation (6) with $q'(X')$ and $\Delta'(X')$ set equal to zero for $X'<X$. It then follows immediately that

$$\frac{\partial}{\partial X}\phi(\beta b,X)=-\varepsilon\Big(q'(X)\frac{\overline{\delta m}}{\delta\beta b(X)}-\Delta'(X)\overline{\{m\}_X^2}\Big) \tag{32}$$

with

$$\varepsilon=(\beta J)^2/2. \tag{33}$$

The first term is obtained as usual by expanding with respect to $\int_X^{X+\delta}\beta b(X)\,\mathrm{d}X$ up to second order, the second term simply by differentiating the integrals at the lower bound. $\{m\}_X$ has not to be differentiated because of stationarity. Since X is the longest timescale that occurs, we can replace $\delta/\delta\beta b(X)$ by $\partial/\partial\beta b$. Thus we obtain

$$\frac{\partial}{\partial X}\phi(Y,X)=-\varepsilon\Big[q'(X)\frac{\partial^2}{\partial Y^2}\phi(Y,X)-\Delta'(X)\Big(\frac{\partial}{\partial Y}\phi(Y,X)\Big)^2\Big] \tag{34}$$

with the boundary condition

$$\phi(Y,1)=\log(2\cosh Y). \tag{35}$$

Equations (37) and (38) can be converted into an integral equation which can be iterated. The iteration yields the expansion along the de Almeida–Thouless line.

In this way we expanded $\phi(Y,0)$ up to third order:

$$\phi(Y,0)=\log(2\cosh Y)+\varepsilon(\Delta(0)+f(0)m')+\varepsilon^2\Big(\frac{f(0)^2}{2!}m'''+2\int_0^1\mathrm{d}X\ \Delta'(X)f(X)m'^2\Big)$$
$$+\varepsilon^3\Big(\frac{f(0)^3}{3!}m^{(5)}-8\int_0^1\mathrm{d}X\ \Delta'(X)f(X)^2m^2m'^2$$

$$-8f(0)\int_0^1 \mathrm{d}X\ \Delta'(X)f(X)(1-5m^2)m'^2$$

$$+8\int_0^1 \mathrm{d}X\ \Delta'(X)(\Delta(X)-\Delta(0))f(X)(1-3m^2)m'^2\Big) \tag{36}$$

with

$$f(X)=q(1)-q(X)-\Delta(X) \qquad m=\tanh Y. \tag{37}$$

Variation of the free energy with respect to $q'(X)$ (or $\Delta'(X)$) gives

$$0=\overline{m'^3}\Delta'(X)+2\overline{m^2m'^2}q'(X) \tag{38}$$

according to equation (26) and

$$\Delta(0)=\frac{(1-(\beta J)^2\overline{(n'^2})\overline{m^2m'^2}}{(\beta J)^4\overline{((1-3m^3)m'^2})^2}. \tag{39}$$

Here

$$\overline{m'^2}=\int\frac{\mathrm{d}z}{\sqrt{2\pi}}\mathrm{e}^{-z^2/2}\tanh'(\beta b+\beta Jz\sqrt{q(0)})^2. \tag{40}$$

Note that $\Delta'(X)$ is of higher order than $q'(X)$ for $b\to 0$. $\Delta(0)$ is positive below the transition, and inequality (30) ensures that the denominator in equation (39) does not generate an additional singularity.

We may insert equations (38) and (39) into equations (36) and (31). Then the free energy has to be made stationary with respect to $q(0)$. We get near the de Almeida–Thouless line up to first order

$$q(0)=q_0-f(0) \tag{41}$$

where q_0 is the Sherrington–Kirkpatrick value of the Edwards–Anderson order parameter and

$$f(0)=-\frac{1-(\beta J)^2\overline{m'^2}}{2(\beta J)^4\overline{(1-3m^2)m'^2}}. \tag{42}$$

The result for the free energy is then up to third order

$$-\beta f=-\beta f_0+\frac{1}{12}\frac{(1-(\beta J)^2\overline{m_0'^2})^3\overline{m_0^2m_0'^2}}{(\overline{(1-3m_0^2)m_0'^2})^4(\beta J)^6} \tag{43}$$

The index means that the expressions have to be calculated for the Sherrington–Kirkpatrick solution. The correction is negative in the spin glass phase; thus the free energy is greater than for the Sherrington–Kirkpatrick solution. The transition is of third order for arbitrary external fields.

4. Conclusions

We have derived a number of unique properties of the mean field theories of Parisi and Sompolinski. $q(X)$ must be a monotonic increasing function of X corresponding to an increasing decay of dynamical spin correlations with time. $\Delta(X)$ must be a monotonic decreasing function of X corresponding to increasing renormalisation of the local spin due to interaction. At each timescale there exists a relation corresponding to marginal dynamic stability. We have derived an exact equation for all external fields which replaces the Parisi–Toulouse hypothesis. Using this relation we will derive elsewhere the low-field behaviour of the susceptibility for all temperatures $0 < T < T_c$. The ratio $-\Delta'(X)/q'(X)$, which is equal to Parisi's X, is proportional to the averaged square of the 3-spin vertex and thus proportional to certain self-energy diagrams for the susceptibility which have been neglected in a derivation (Sommers 1978) of the equations of Thouless *et al* (1977), because they are formally of higher order in $1/N$. These diagrams, however, diverge in the spin glass phase. This gives us a hint for deriving a consistent mean field theory on a purely static basis. Furthermore, we have derived a Parisi differential equation for the Sompolinsky free energy functional. The solution has been expanded along the de Almeida–Thouless line for arbitrary external fields. The transition is of third order, in contrast to the prediction of Parisi and Toulouse (1980).

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